Heat Conduction in Lenses (Summary)

We describe **five idealized heat conduction problems** in connection with optical lenses, together with their analytic solution. Full details and derivations can be found in the paper *"Heat Conduction in Lenses"* by Beat Aebischer, May 2004, submitted to the journal "American Mathematical Monthly".

For every problem we provide two **Matlab programs**: one to compute the solution (in dimensionless form) and another one to illustrate the use of the former and to plot sample results. The formula numbers below agree with those in the paper and in the downloadable Matlab programs.

Commonalities of the Problems

An initially uniformly cold lens is heated up by keeping the mounting at a higher temperature or by delivering a constant heat flux through the mounting or by absorption of solar radiation. The lens and its mounting are assumed to be radially symmetric. Except for the last problem, the lens is modeled as a cylinder with cylinder axis equal to the optical axis. We always assume that the lens exchanges no heat through its optical surfaces.

If necessary we average the temperature in the direction of the optical axis. It is then described by the function

(0.2) $u = u(r,t), \quad 0 \le r \le R, \quad t \ge 0$,

where t denotes time, r is the distance from the axis and R is the radius of the lens.

Problem 1: Heating by Holding the Temperature of the Mounting Constant

Problem 2: Prescribed Heat Flux for some Time

Problem 3: Heating only Part of the Cylinder Mantle

Problem 4: Heating by Absorption of Radiation

Problem 4s: Stationary Case for a Lens of Varying Thickness

Problem 1: Heating by Holding the Temperature of the Mounting Constant

We denote the **initial temperature of the lens** by u_0 and the **temperature of the mounting** by u_m. There is a layer of **glue** of **thickness** d and **thermal conductivity** λ_{glue} between the glass lens and the metal mounting. With the quantities [SI units in brackets]

$$\kappa = \frac{\lambda}{c\rho}: \text{ heat conductivity } [\text{m}^2\text{s}^{-1}],$$
$$\lambda: \text{ thermal conductivity } [\text{W m}^{-1}\text{K}^{-1}],$$
$$c: \text{ specific heat } [\text{J kg}^{-1}\text{K}^{-1}],$$
$$\rho: \text{ density } [\text{kg m}^{-3}]$$

all referring to the glass and with

(1.3) $$D = d \cdot \frac{\lambda}{\lambda_{\text{glue}}},$$

the temperature u is determined by the following partial differential equation:

(1.1a) $$\frac{\partial u}{\partial t} = \kappa \cdot \Delta u = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right],$$

(1.1b) $$u(r,0) = u_0,$$

(1.1c) $$D \cdot \frac{\partial u}{\partial r}(R,t) + u(R,t) - u_m = 0.$$

With the **scaling**

(1.4) $$\tilde{r} = \frac{r}{R}, \quad T = \frac{R^2}{\kappa}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{D} = \frac{D}{R} = \frac{d}{R}\frac{\lambda}{\lambda_{\text{glue}}}, \quad \tilde{u}(\tilde{r},\tilde{t}) = \frac{u(r,t) - u_m}{u_m - u_0}$$

we get the equivalent **dimensionless problem**

(1.5a) $$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}},$$

(1.5b) $$\tilde{u}(\tilde{r},0) = -1,$$

(1.5c) $$\tilde{D} \cdot \frac{\partial \tilde{u}}{\partial \tilde{r}}(1,\tilde{t}) + \tilde{u}(1,\tilde{t}) = 0.$$

Solution:

(1.22) $$\tilde{u}(\tilde{r},\tilde{t}) = \sum_{j=1}^{\infty} \frac{-2\, e^{-k_j^2 \tilde{t}}}{(\tilde{D}^2 k_j^2 + 1) \cdot k_j J_1(k_j)} J_0(k_j \tilde{r}),$$

where J_0 and J_1 are the Bessel functions of the first kind, and the constants $0 < k_1 < k_2 < \ldots$ are determined by the equation $J_0(k_j) - \tilde{D} \cdot k_j \cdot J_1(k_j) = 0$.

Matlab programs:

HeatProblem1.m Solution of (1.5a-c), using (1.22). Download program

HeatFigure1_1.m Uses HeatProblem1.m to produce a sample plot of results. Download program

Problem 2: Prescribed Heat Flux for some Time

Denoting by q [W/m^2] the **heat flux density** at the interface between the glass and the glue, which is turned on from time 0 to time t_0, we have the following problem:

(2.1a) $$\frac{\partial u}{\partial t} = \kappa \cdot \Delta u = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right],$$

(2.1b) $$u(r,0) = u_0 ,$$

(2.1c) $$\lambda \cdot \frac{\partial u}{\partial r}(R,t) = q \cdot H(t_0 - t) ,$$

where H denotes the Heaviside function (the characteristic function of the positive reals).

With the **scaling**

(2.2) $$\tilde{r} = \frac{r}{R}, \quad T = \frac{R^2}{\kappa}, \quad \tilde{t} = \frac{t}{T} \quad \tilde{t}_0 = \frac{t_0}{T}, \quad U = \frac{qR}{\lambda}, \quad \tilde{u}(\tilde{r},\tilde{t}) = \frac{u(r,t) - u_0}{U}$$

we get the **dimensionless problem**

(2.3a) $$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}} ,$$

(2.3b) $$\tilde{u}(\tilde{r},0) = 0 ,$$

(2.3c) $$\frac{\partial \tilde{u}}{\partial \tilde{r}}(1,\tilde{t}) = H(\tilde{t}_0 - \tilde{t}) .$$

Solution:

(2.20) $$\tilde{u}(\tilde{r},\tilde{t}) = \begin{cases} \dfrac{\tilde{r}^2}{2} + 2\tilde{t} - \dfrac{1}{4} - \displaystyle\sum_{j=1}^{\infty} \frac{2e^{-k_j^2 \tilde{t}}}{k_j^2 J_0(k_j)} J_0(k_j \tilde{r}), & 0 \le \tilde{t} \le \tilde{t}_0 \\ 2\tilde{t}_0 + \displaystyle\sum_{j=1}^{\infty} \frac{2}{k_j^2 J_0(k_j)} \left(1 - e^{-k_j^2 \tilde{t}_0}\right) e^{-k_j^2 (\tilde{t} - \tilde{t}_0)} J_0(k_j \tilde{r}), & \tilde{t} > \tilde{t}_0 \end{cases}$$

where the k_j are the positive zeros of the Bessel function J_1 in ascending order.

Matlab programs:

HeatProblem2.m   Solution of (2.3a-c), using (2.20). Download program
HeatFigure2_1.m   Uses HeatProblem2.m to produce a sample plot of results. Download program

Problem 3: Heating only Part of the Cylinder Mantle

The lens is a cylinder of radius R and height h_L. Part of the cylinder mantle, a ring of width h_m, is kept at temperature u_m, another part, a ring of width h_h, delivers a constant heat flux density q [W/m^2] from time 0 to t_0 and then this part of the boundary is thermally isolated. The remaining mantle surface (of width $h_L - h_h - h_m \geq 0$) and all other surfaces of the lens are always thermally isolated.

The average temperature $u(r,t)$ along the axis direction is determined by:

(3.1a) $$\frac{\partial u}{\partial t} = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right],$$

(3.1b) $$u(r,0) = u_0 \,,$$

(3.1c) $$d\,\frac{\lambda}{\lambda_{\text{glue}}}\frac{h_L}{h_m}\cdot\frac{\partial u}{\partial r}(R,t) + u(R,t) - \left(u_m + H(t_0 - t)\cdot q\frac{h_h}{h_m}\frac{d}{\lambda_{\text{glue}}} \right) = 0\,.$$

With the **scaling**

(3.4) $$\tilde r = \frac{r}{R},\quad T = \frac{R^2}{\kappa},\quad \tilde t = \frac{t}{T},\quad \tilde t_0 = \frac{t_0}{T},\quad \tilde D = \frac{d}{R}\frac{\lambda}{\lambda_{\text{glue}}}\frac{h_L}{h_m},\quad \tilde u(\tilde r,\tilde t) = \frac{u(r,t)-u_m}{u_e - u_0},\quad \gamma = \frac{u_e - u_m}{u_e - u_0},$$

where

(3.3) $$u_e := u_m + q\frac{h_h}{h_m}\frac{d}{\lambda_{\text{glue}}}$$

we have the **dimensionless problem**

(3.5a) $$\frac{\partial \tilde u}{\partial \tilde t} = \frac{\partial^2 \tilde u}{\partial \tilde r^2} + \frac{1}{\tilde r}\frac{\partial \tilde u}{\partial \tilde r},$$

(3.5b) $$\tilde u(\tilde r,0) = \gamma - 1,$$

(3.5c) $$\tilde D \cdot \frac{\partial \tilde u}{\partial \tilde r}(1,\tilde t) + \tilde u(1,\tilde t) = \gamma \cdot H(\tilde t_0 - \tilde t)\,.$$

Solution:

(3.12) $$\tilde u(\tilde r,\tilde t) = \begin{cases} \gamma - \displaystyle\sum_{j=1}^{\infty} \frac{2e^{-k_j^2 \tilde t}}{(\tilde D^2 k_j^2 + 1)k_j J_1(k_j)} J_0(k_j \tilde r), & 0 \leq \tilde t \leq \tilde t_0 \\ \displaystyle\sum_{j=1}^{\infty} \frac{2}{(\tilde D^2 k_j^2 + 1)k_j J_1(k_j)} \left(\gamma e^{-k_j^2(\tilde t - \tilde t_0)} - e^{-k_j^2 \tilde t} \right) J_0(k_j \tilde r), & \tilde t \geq \tilde t_0 \end{cases}$$

with the constants $0 < k_1 < k_2 < \ldots$ determined by $J_0(k_j) - \tilde D \cdot k_j \cdot J_1(k_j) = 0$.

Matlab programs:

HeatProblem3.m Solution of (3.5a-c), using (3.12). Download program

HeatFigure3_2.m Uses HeatProblem3.m to produce a sample plot of results. Download program

Problem 4: Heating by Absorption of Radiation

The lens is still a cylinder of radius R and thickness h. The mounting, which is attached to the cylinder mantle by a glue layer of thickness d, is kept at a constant temperature u_0. But now one of the optical surfaces is **coated** and absorbs solar radiation. We model this effect by a constant and homogeneous heat flux density q [W/m^2], which is turned on from time 0 to t_0 and then is turned off. The other optical surface of the lens is thermally isolated.

The average temperature $u(r,t)$ along the axis direction is determined by:

(4.1a) $$\frac{\partial u}{\partial t} = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{q}{\lambda h} H(t_0 - t) \right],$$

(4.1b) $$u(r,0) = u_0,$$

(4.1c) $$D \cdot \frac{\partial u}{\partial r}(R,t) + u(R,t) = u_0$$

with

(4.2) $$D := d \frac{\lambda}{\lambda_{\text{glue}}}.$$

With the **scaling**

(4.3) $$\tilde{r} = \frac{r}{R}, \quad T = \frac{R^2}{\kappa}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{t}_0 = \frac{t_0}{T}, \quad \tilde{D} = \frac{D}{R}, \quad \tilde{u}(\tilde{r},\tilde{t}) = \frac{u(r,t) - u_0}{U}, \quad U = \frac{qR}{\lambda}, \quad \tilde{q} = \frac{R}{h}$$

the **dimensionless problem** becomes

(4.4a) $$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}} + \tilde{q} \cdot H(\tilde{t}_0 - \tilde{t}),$$

(4.4b) $$\tilde{u}(\tilde{r},0) = 0,$$

(4.4c) $$\tilde{D} \cdot \frac{\partial \tilde{u}}{\partial \tilde{r}}(1,\tilde{t}) + \tilde{u}(1,\tilde{t}) = 0.$$

Solution:

(4.19) $$\tilde{u}(\tilde{r},\tilde{t}) = \begin{cases} \dfrac{\tilde{q}}{4}(1 + 2\tilde{D} - \tilde{r}^2) - \displaystyle\sum_{j=1}^{\infty} \frac{2\tilde{q} e^{-k_j^2 \tilde{t}}}{(\tilde{D}^2 k_j^2 + 1) k_j^3 J_1(k_j)} J_0(k_j \tilde{r}), & 0 \le \tilde{t} \le \tilde{t}_0 \\ \displaystyle\sum_{j=1}^{\infty} \frac{2\tilde{q}}{(\tilde{D}^2 k_j^2 + 1) k_j^3 J_1(k_j)} \left(e^{-k_j^2(\tilde{t} - \tilde{t}_0)} - e^{-k_j^2 \tilde{t}} \right) J_0(k_j \tilde{r}), & \tilde{t} \ge \tilde{t}_0 \end{cases}$$

with the constants $0 < k_1 < k_2 < \ldots$ determined by $J_0(k_j) - \tilde{D} \cdot k_j \cdot J_1(k_j) = 0$.

Matlab programs:

HeatProblem4.m Solution of (4.4a-c), using (4.19). Download program

HeatFigure4_1.m Uses HeatProblem4.m to produce a sample plot of results. Download program

Problem 4s: Stationary Case for a Lens of Varying Thickness

Same as problem 4, but the heat flux is never turned off ($t_0 = \infty$) and the thickness h of the lens now varies according to $h(r) = h_0 [1 - c\,(r/R)^2]$. This approximates a biconvex lens with the index of refraction n and with optical power

(4.44) $$\frac{1}{f} = \frac{2(n-1)}{R^2} h_0 c \,.$$

The stationary solution $u(r)$ for the average temperature along the axis direction is determined by the ordinary differential equation

$$u'' + \frac{u'}{r} + \frac{q}{\lambda h(r)} = 0 \,,$$

$$D \cdot \frac{du}{dr}(R) + u(R) = u_0$$

with D from (4.2) above.

Scaling with (4.3) and with $\tilde{q}$ generalized to

(4.27) $$\tilde{q}(\tilde{r}) := \frac{R}{h(R\tilde{r})} = \frac{R}{h_0(1 - c\tilde{r}^2)} =: \frac{1}{\tilde{h}_0(1 - c\tilde{r}^2)}$$

yields the **dimensionless problem**

(4.28) $$\tilde{u}'' + \frac{\tilde{u}'}{\tilde{r}} + \tilde{q}(\tilde{r}) = 0 \,,$$

(4.29) $$\tilde{D} \cdot \tilde{u}'(1) + \tilde{u}(1) = 0 \,.$$

Analytic Solution:

(4.37) $$\tilde{u}(\tilde{r}) = \frac{1}{4\tilde{h}_0 c} \int_c^{c\tilde{r}^2} \frac{\log(1-x)}{x} dx - \frac{\tilde{D}}{2\tilde{h}_0 c} \log(1-c) \,.$$

Matlab programs:

HeatProblem4s.m	Solution of (4.28/29), using (4.37). Download program
HeatFigure4_2.m	Uses HeatProblem4s.m to produce a sample plot of results. Download program